HOW TO ROTATE A VECTOR

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ABSTRACT
Rotation of a vector around a direction in space is shown.

Key words: vectors — vector analysis — Euclidean geometry

1. ROTATION OF A VECTOR.

We wish to rotate a vector $\vec{v}$ counterclockwise around an axis $\vec{w}$ by an angle φ. The geometric picture is illustrated in Figure 1. The components of $\vec{v}$ and $\vec{v}'$ perpendicular to $\vec{w}$ are $\vec{u}$ and $\vec{u}'$. We have

$$\vec{u} = \vec{v} - (\vec{v}\cdot\hat{w})\,\hat{w} \tag{1}$$

and

$$\vec{u}' = \vec{v}' - (\vec{v}\cdot\hat{w})\,\hat{w} \tag{2}$$

since it is apparent that $\vec{v}\cdot\hat{w} = \vec{v}'\cdot\hat{w}$.

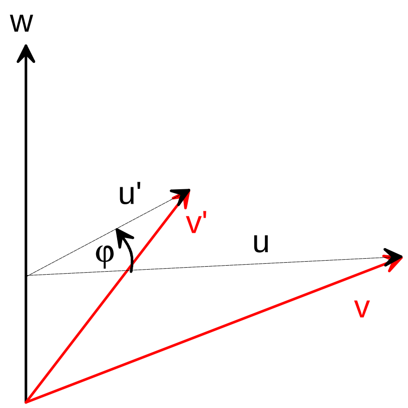

Figure 1

The trick to determining $\vec{v}'$ is to notice that the rotation of the vector is just a coordinate rotation in the $\vec{u}\,\vec{u}'$ plane. The unit vector perpendicular to $\vec{u}$ is $\frac{\hat{w}\times\vec{u}}{\|\hat{w}\times\vec{u}\|}$. Hence we can write $\vec{u}'$ in terms of its components parallel and perpendicular to $\vec{u}$,

$$\vec{u}' = u\left(\hat{u}\cos\varphi + \tfrac{\hat{w}\times\vec{u}}{\|\hat{w}\times\vec{u}\|}\sin\varphi\right) \tag{3}$$

But $\|\hat{w}\times\vec{u}\| = \sqrt{u^2 - (\vec{u}\cdot\hat{w})^2} = u$, so

$$\vec{u}' = \vec{u}\cos\varphi - \vec{u}\times\hat{w}\sin\varphi \tag{4}$$

Now, from (2) we have

$$\vec{v}' = \vec{u}' + (\vec{v}\cdot\hat{w})\,\hat{w} \tag{5}$$

Using (4) for $\vec{u}'$ and (1) for $\vec{u}$, this becomes

$$\begin{aligned}\vec{v}' &= \vec{u}\cos\varphi - \vec{u}\times\hat{w}\sin\varphi + (\vec{v}\cdot\hat{w})\,\hat{w}\\ &= [\vec{v} - (\vec{v}\cdot\hat{w})\,\hat{w}]\cos\varphi\\ &\quad - [\vec{v} - (\vec{v}\cdot\hat{w})\,\hat{w}]\times\hat{w}\sin\varphi\\ &\quad + (\vec{v}\cdot\hat{w})\,\hat{w}\end{aligned} \tag{6}$$

Hence we have the result

$$\boxed{\vec{v}' = \vec{v}\cos\varphi + (1-\cos\varphi)(\vec{v}\cdot\hat{w})\,\hat{w} - \vec{v}\times\hat{w}\sin\varphi} \tag{7}$$

2. ROTATION OF A POSITION VECTOR AROUND A POINT IN SPACE.

Now consider a position vector $\vec{r}$ relative to a coordinate origin O. We wish to rotate this position vector counterclockwise by and angle φ around an axis $\vec{w}$ with anchor point $\vec{p}$. See Figure 2.

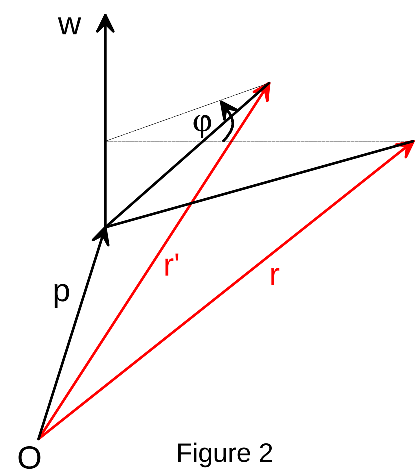

Figure 2

Notice that eq. (7) can be viewed as a linear operator, $\vec{R}_{\hat{w},\varphi}(\vec{v})$, acting on the argument $\vec{v}$. That is, let

$$\vec{R}_{\hat{w},\varphi}(\vec{v}) \equiv \vec{v}\cos\varphi + (1-\cos\varphi)(\vec{v}\cdot\hat{w})\,\hat{w} - \vec{v}\times\hat{w}\sin\varphi \tag{8}$$

Then it is apparent from Figure 2 that the rotated position vector can be written

$$\boxed{\vec{r}' = \vec{p} + \vec{R}_{\hat{w},\varphi}(\vec{r}-\vec{p})} \tag{9}$$